Comput Econ (2007) 29:151–158
DOI 10.1007/s10614-006-9069-8

Cutting the hedge

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Received: 5 October 2006/Accepted: 12 October 2006 /
Published online: 30 January 2007

1 Introduction

Do you enjoy chores such as mowing the lawn or, as it is called in Canada, shovelling the snow? Below we discuss a simpler method of trimming the hedge, suggested by Barone-Adesi, Engle and Mancini (2004). Assuming the option price is homogeneous our calculation is model independent and provides delta hedge ratios immediately from market data. For strike prices which are close enough our approximation will work for moderate departures from homogeneity. Indeed, independently of the model the continuity of option price as a function of asset price and strike price ensures that for c close to 1 $C(cS_0, cK)$ equals approximately $cC(S_0, K)$. In fact we do not need to impose homogeneity everywhere, but only as a local approximation. For common smiles found empirically in SP index options our approximation holds well if the ratio of the increment in strike price over the asset price is less than 0.05, (in practice this ratio is around $10/1000 = 0.01$).

The assumption of homogeneity is strong. When the logarithm of $S(t)$ follows a general Levy process, under the risk-neutral measure, then $C(S, K)$ is homogeneous (of degree 1), though there is also a volatility smile/skew. However, our method applies, and provides a very easy calculation of the delta from observed data. Of course, for general Levy processes the delta $\mathrm{d}C/\mathrm{d}S$ does not provide a complete hedge, as the market is incomplete, (unless $S(t)$ is just a Brownian

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motion or a single Poisson process with a drift). However, delta $\mathrm{d}C/\mathrm{d}S$ is a useful quantity. Euler's formula

$$C = S\mathrm{d}C/\mathrm{d}S + K\mathrm{d}C/\mathrm{d}K$$

is a consequence of the homogeniety of $C(S, K)$ and this has been used to determine delta. However, our method is different.

Hedge ratios are important in asset management. They are approximated as the ratio of the change in an option price divided by the change in the underlying. However, only one underlying price is observed today, together with its associated option prices. We determine option prices for different values of the underlying by considering prices for different strikes and using the homogeneity of the option price.

2 Notation

Suppose S_0 is the price today of the underlying asset and S_T is the (unknown) price at time T, the expiration time of the option.

Consider a strike price K and suppose the continuously compounded risk free rate is r. With E denoting the expectation under the risk neutral measure the price today of a European call is

$$C(S_0, K) = E\big[\mathrm{e}^{-rT}(S_T - K)^* | S_0\big].$$

Our discussion also applies to puts. We assume that for $c > 0, C(cS_0, cK) = cC(S_0, K)$. This is certainly the case in the Black–Scholes case.

The usual hedge requires the evaluation of the delta

$$\frac{\partial C}{\partial S}(S_0, K).$$

For example, in the Black–Scholes framework (Elliott & Kopp, 1999),

$$\frac{\partial C}{\partial S}(S_0, K) = \Phi(d_1)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{y^2/2}\,\mathrm{d}y$ is the standard normal distribution and

$$d_1 = \frac{\ell\mathrm{n}\,(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

Here σ is the volatility of S which has to be estimated.

The Black–Scholes model assumes σ is constant. However, observed market prices imply that σ should possibly vary with K. This gives rise to the volatility 'smile' when the implied value of σ is graphed against K.

The delta is often approximated as

$$\frac{\Delta C}{\Delta S} = \frac{C(S_0 + \Delta S, K) - C(S_0, K)}{(S_0 + \Delta S) - S_0} .$$

For example, this approximation is used in binomial tree models.

3 Homogeneity

Today there is only one price S_0 given in the market. However, the market often quotes prices for different strikes K. We shall use the homogeneity of the option price to deduce implied option prices for different values of S_0.

Suppose the option prices $C(S_0, K_1)$, $C(S_0, K)$ and $C(S_0, K_2)$ are quoted today in the market and $K_1 < K < K_2$.

In fact suppose $K_1 = \alpha K$ and $K_2 = \beta K$. Now the option price is assumed homogeneous in S_0 and K. That is, for $c > 0$

$$C(cS_0, cK) = cC(S_0, K).$$

Therefore, for the observed prices

$$C(S_0, K_1) \quad \text{and} \quad C(S_0, K_2)$$

we have

$$C(S_0, \alpha K) = \alpha C(\alpha^{-1} S_0, K)$$
$$C(S_0, \beta K) = \beta C(\beta^{-1} S_0, K).$$

Write

$$\alpha^{-1} S_0 = S_0 + \Delta_1 S$$
$$\beta^{-1} S_0 = S_0 + \Delta_2 S$$

so

$$C(S_0 + \Delta_1 S, K) = \alpha^{-1} C(S_0, \alpha K)$$
$$C(S_0 + \Delta_2 S, K) = \beta^{-1} C(S_0, \beta K)$$

where the quantities on the right are available from market prices. To compensate for any ‘smile’ effect we calculate ratios $\Delta C/\Delta S$ for strikes, K_1, K_2 above and below K.

Write

$$\Delta_K^- = \frac{C(S_0 + \Delta_1 S, K) - C(S_0, K)}{(S_0 + \Delta_1 S) - S_0}$$

$$= \frac{\alpha^{-1} C(S_0, K_1) - C(S_0, K)}{\alpha^{-1} S_0 - S_0}$$

$$\Delta_K^+ = \frac{C(S_0 + \Delta_2 S, K) - C(S_0, K)}{(S_0 + \Delta_2 S) - S_0}$$

$$= \frac{\beta^{-1} C(S_0, K_2) - C(S_0, K)}{\beta^{-1} S_0 - S_0} .$$

Our estimate for delta is then

$$\Delta_K := \frac{\Delta_K^- + \Delta_K^+}{2} .$$

By considering the upper and lower estimates, Δ_K^- and Δ_K^+, and then averaging the 'smile' effect should be reduced.

4 Example

Suppose the price today of the underlying is $S_0 = 180$.

Further, suppose the prices of 6 month options with strikes $210, 220$ and 230 are quoted as:

$$C(180, 210) = 6.75$$
$$C(180, 220) = 4.75$$
$$C(180, 230) = 3.29.$$

Using the homogeneity of the option price we deduce

$$C(188.57, 220) = \frac{220}{210}\, C(180, 210) = 7.07$$

and

$$C(172.17, 220) = \frac{220}{230}\, C(180, 230) = 3.15.$$

Therefore,

$$\Delta_{220}^{-} = \frac{C(172.17, 220) - C(180, 220)}{172.17 - 180}$$

$$= \frac{4.75 - 3.15}{7.83} = 0.204$$

and

$$\Delta_{220}^{+} = \frac{C(188.57, 220) - C(180, 220)}{188.57 - 180}$$
$$= 0.271.$$

Consequently an approximate value for the delta is

$$\Delta_{220} = \frac{\Delta_{220}^{-} + \Delta_{220}^{+}}{2} = 0.238.$$

The above option prices are given by Black–Scholes with $T = 0.5, r = 5\%$ and $\sigma = 30\%$. The Black–Scholes delta is then 0.23 so our approximation is reasonably accurate.

We now provide two sets of data (Appendix Tables 1, 2) in which our method is used to calculate the hedging ratios. In the first we compute the Black–Scholes prices associated with a volatility smile. The volatility, sigma, takes values from 0.15 down to 0.11 and then increases again to 0.14. These values are associated with strikes X which take values from 900 to 1100 based on a constant underlying price of 1000. With an expiration of 3 months, 0.25 of a year, the Black–Scholes call prices are calculated as well as the $N(d_1)$ from the Black–Scholes model. The approximate hedging ratio H of this paper is also found and the error between H and $N(d_1)$ given.

The two graphs (Figs. 1, 2) represent the volatility smile and the error in the hedge ratio. We note that the error between the Black–Scholes delta $N(d_1)$ and our approximate delta is negative when the volatility is decreasing and positive when the volatility is increasing. For the given values the hedge ratio reaches a maximum at the strike $X = 1030$. The final column of values gives the butterfly spread value and we see that when the strike is 1030 the butterfly spread has a negative price, which represents an arbitrage opportunity. Therefore our hedge ratio, based on option prices, appears to deteriorate when these prices allow for arbitrage opportunities. However, the Black–Scholes model is itself an approximation and the hedge ratios given by $N(d_1)$ are approximations.

The second set of figures provides Black–Scholes hedge ratios $N(d_1)$ and our corresponding approximate values H when the volatility is constant, that is when there is no smile. In this case the errors are much smaller and constant.

Fig. 1 Volatility Smile

0.0000	0.1500
10.0000	0.1400
20.0000	0.1350
30.0000	0.1300
40.0000	0.1250
50.0000	0.1200
60.0000	0.1150
70.0000	0.1120
80.0000	0.1100
90.0000	0.1100
100.0000	0.1120
110.0000	0.1150
120.0000	0.1150
130.0000	0.1200
140.0000	0.1220
150.0000	0.1250
160.0000	0.1270
170.0000	0.1300
180.0000	0.1320
190.0000	0.1350

Fig. 2 Delta Error

10	-0.04635712
20	-0.037662436
30	-0.044145479
40	-0.051433799
50	-0.059454676
60	-0.052460339
70	-0.037286813
80	-0.015165086
90	0.020018894
100	0.048986563
110	0.029705333
120	0.046086335
130	0.066630216
140	0.042604593
150	0.040903934
160	0.035701882
170	0.033844371
180	0.028626893
190	0.041151685

Acknowledgements The authors are grateful to Peter Carr, Emanuel Derman, Farshid Jamshidian and John van der Hoek for their helpful comments. The financial support of FinRisk, a network of the Swiss National Fund, is gratefully acknowledged.

Appendix

This is an example of an artificial smile, calibrated at values typical of the standard and poor index, to highlight the differences of our approximate hedging from Black–Scholes' Deltas

Table 1 Hedging the smile

Sigma	Time	Stock price	$N(d1)$	X	Call	Approx. H. ratio	Error	Butterfly (convexity)
0.15	0.2500	1000.00	0.9322	900.0000	105.6729			
0.14	0.2500	1000.00	0.9245	910.0000	95.8443	0.9708	-0.0464	0.42642727

Table 1 continued

Sigma	Time	Stock price	$N(d1)$	X	Call	Approx. H. ratio	Error	Butterfly (convexity)
0.135	0.2500	1000.00	0.9073	920.0000	86.4423	0.9450	−0.0377	0.13970098
0.13	0.2500	1000.00	0.8862	930.0000	77.1799	0.9304	−0.0441	0.17671892
0.125	0.2500	1000.00	0.8602	940.0000	68.0942	0.9116	−0.0514	0.22349863
0.12	0.2500	1000.00	0.8283	950.0000	59.2321	0.8877	−0.0595	0.2821582
0.115	0.2500	1000.00	0.7893	960.0000	50.6521	0.8417	−0.0525	0.6790233
0.112	0.2500	1000.00	0.7385	970.0000	42.7511	0.7758	−0.0373	0.68677821
0.11	0.2500	1000.00	0.6783	980.0000	35.5369	0.6935	−0.0152	1.00095304
0.11	0.2500	1000.00	0.6097	990.0000	29.3236	0.5896	0.0200	1.10712013
0.112	0.2500	1000.00	0.5378	1000.0000	24.2175	0.4888	0.0490	0.91978495
0.115	0.2500	1000.00	0.4685	1010.0000	20.0311	0.4388	0.0297	0.08067937
0.115	0.2500	1000.00	0.4011	1020.0000	15.9255	0.3550	0.0461	1.56190898
0.12	0.2500	1000.00	0.3445	1030.0000	13.3817	0.2779	0.0666	−0.04868899
0.122	0.2500	1000.00	0.2908	1040.0000	10.7893	0.2482	0.0426	0.61900727
0.125	0.2500	1000.00	0.2453	1050.0000	8.8158	0.2044	0.0409	0.22181232
0.127	0.2500	1000.00	0.2042	1060.0000	7.0642	0.1685	0.0357	0.45790977
0.13	0.2500	1000.00	0.1709	1070.0000	5.7705	0.1370	0.0338	0.13396008
0.132	0.2500	1000.00	0.1409	1080.0000	4.6107	0.1123	0.0286	0.32563952
0.135	0.2500	1000.00	0.1175	1090.0000	3.7766	0.0764	0.0412	0.33598334
0.14	0.2500	1000.00	0.1015	1100.0000	3.2785			

The error pattern reflects the smile pattern. It is negative for declining volatility, positive otherwise. It reaches a maximum at strike 1030. At that price there is a violation of convexity that leads to an arbitrage opportunity (the butterfly spread has a negative price). The Black–Scholes hedge ratios calibrated at the implied volatilities are in any case approximations, the true model being unknown

Table 2 Hedging the smile

Sigma	Time	Stock Price	$N(d1)$	X	Call	Approx. H. ratio	Error	Butterfly (convexity)
0.2	0.2500	1000.00	0.8731	900.0000	109.9760			
0.2	0.2500	1000.00	0.8486	910.0000	101.6298	0.8482	0.0004	0.28314734
0.2	0.2500	1000.00	0.8216	920.0000	93.5669	0.8212	0.0004	0.3081619
0.2	0.2500	1000.00	0.7920	930.0000	85.8120	0.7916	0.0003	0.33117454
0.2	0.2500	1000.00	0.7600	940.0000	78.3884	0.7597	0.0003	0.35157447
0.2	0.2500	1000.00	0.7259	950.0000	71.3163	0.7256	0.0002	0.3688285
0.2	0.2500	1000.00	0.6900	960.0000	64.6130	0.6898	0.0002	0.38250445
0.2	0.2500	1000.00	0.6525	970.0000	58.2923	0.6524	0.0001	0.39228853
0.2	0.2500	1000.00	0.6139	980.0000	52.3638	0.6139	0.0001	0.39799571
0.2	0.2500	1000.00	0.5746	990.0000	46.8333	0.5746	0.0000	0.39957287
0.2	0.2500	1000.00	0.5349	1000.0000	41.7024	0.5349	−0.0001	0.39709502
0.2	0.2500	1000.00	0.4952	1010.0000	36.9686	0.4953	−0.0001	0.39075528
0.2	0.2500	1000.00	0.4560	1020.0000	32.6256	0.4562	−0.0002	0.3808497
0.2	0.2500	1000.00	0.4176	1030.0000	28.6634	0.4178	−0.0002	0.36775844
0.2	0.2500	1000.00	0.3803	1040.0000	25.0690	0.3806	−0.0003	0.35192454
0.2	0.2500	1000.00	0.3444	1050.0000	21.8264	0.3448	−0.0003	0.3338319
0.2	0.2500	1000.00	0.3102	1060.0000	18.9178	0.3106	−0.0004	0.31398365
0.2	0.2500	1000.00	0.2779	1070.0000	16.3231	0.2783	−0.0004	0.29288219
0.2	0.2500	1000.00	0.2476	1080.0000	14.0212	0.2480	−0.0004	0.27101154
0.2	0.2500	1000.00	0.2194	1090.0000	11.9904	0.2198	−0.0004	0.24882282
0.2	0.2500	1000.00	0.1934	1100.0000	10.2085			

References

Barone-Adesi, G., Engle, R., & Mancini, L. (2004). *GARCH options in incomplete markets*, Manuscript, Lugano.
Elliott, R. J., & Kopp, P. E. (1999). *Mathematics of financial markets*. Springer-Verlag, (Second edition 2004).